

Patterns in column strict fillings of rectangular arrays.

Johannes Harmse

Department of Mathematics
University of California, San Diego
La Jolla, CA 92093-0112. USA
jharmse@math.ucsd.edu

Jeffrey Remmel*

Department of Mathematics
University of California, San Diego
La Jolla, CA 92093-0112. USA
remmel@math.ucsd.edu

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Abstract

In this paper, we study pattern matching in the set $\mathcal{F}_{n,k}$ of fillings of the $k \times n$ rectangle with the integers $1, \dots, kn$ such that the elements in any column increase from bottom to top. Let P be a column strict tableau of shape 2^k . We say that a filling $F \in \mathcal{F}_{n,k}$ has P -match starting at i if the elements of F in columns i and $i+1$ have the same relative order as the elements of P . We compute the generating functions for the distribution of P -matches and nonoverlapping P -matches for various classes of standard tableaux of shape 2^k . We say that a filling $F \in \mathcal{F}_{n,k}$ is P -alternating if there are P -matches of F starting at all odd positions but there are no P -matches of F starting at even positions. We also compute the generating functions for P -alternating elements of $\mathcal{F}_{n,k}$ for various classes of standard tableaux of shape 2^k .

1 Introduction

Let $\mathcal{F}_{n,k}$ denote the set of all fillings of a $k \times n$ rectangular array with the integers $1, \dots, kn$ such that the elements increase from bottom to top in each column. We let (i, j) denote the cell in i -th row from the bottom and j -th column from the left of the $k \times n$ rectangle and we let $F(i, j)$ denote the element in cell (i, j) of $F \in \mathcal{F}_{n,k}$. For example, the elements of $\mathcal{F}_{2,3}$ are pictured below.

It is easy to see that

$$|\mathcal{F}_{n,k}| = \frac{(kn)!}{(k!)^n}. \quad (1)$$

That is, for each $F \in \mathcal{F}_{n,k}$, allowing all permutations of the elements in each column gives rise to $(k!)^n$ fillings of the $k \times n$ rectangle with the numbers $1, \dots, kn$. Since there are $(kn)!$ fillings of the $k \times n$ rectangle with the numbers $1, \dots, kn$, (1) easily follows.

Given a partition $\lambda = (\lambda_1, \dots, \lambda_k)$ where $0 < \lambda_1 \leq \dots \leq \lambda_k$, we let F_λ denote the Ferrers diagram of λ , i.e. F_λ is the set of left-justified rows of squares where the size of the i -th row, reading from top to bottom, is λ_i . Thus a $k \times n$ rectangular array corresponds to the Ferrers diagram corresponding to n^k . If $F \in \mathcal{F}_{n,k}$ and the integers are increasing in each row, reading from left to right, then F is a standard tableau of shape n^k . We let St_{n^k} denote the set of all

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Figure 1: The elements of $\mathcal{F}_{2,3}$.

standard tableaux of shape n^k and let $st_{n^k} = |St_{n^k}|$. One can use the Frame-Robinson-Thrall hook formula [9] to show that

$$st_{n^k} = \frac{(kn)!}{\prod_{i=0}^{k-1} (i+n) \downarrow_n} \quad (2)$$

where $(n) \downarrow_0 = 1$ and $(n) \downarrow_k = n(n-1) \cdots (n-k+1)$ for $k > 0$.

The goal of this paper is to study pattern matching conditions in $\mathcal{F}_{n,k}$. Clearly, when $k = 1$, we have $\mathcal{F}_{n,1} = S_n$, where S_n is the symmetric group, so our results can be viewed as generalizations of results on pattern matching in S_n . Our long term goal is to extend the work of this paper to the study pattern matching conditions in standard tableaux of rectangular shape as well as pattern matching conditions in column strict tableaux of rectangular shapes over a fixed alphabet $A = \{1, \dots, m\}$. The study of pattern matching conditions in $\mathcal{F}_{n,k}$ is considerably easier than the study of pattern matching conditions in standard tableaux or column strict tableaux. Nevertheless, we shall see that the study of pattern matching conditions in $\mathcal{F}_{n,k}$ requires us to prove some interesting results on pattern matching conditions in standard tableaux. In particular, we shall generalize two classical results on permutations. Given a permutation $\sigma = \sigma_1 \dots \sigma_n \in S_n$, we let

$$\begin{aligned} Rise(\sigma) &= \{i : \sigma_i < \sigma_{i+1}\} \text{ and} \\ Des(\sigma) &= \{i : \sigma_i > \sigma_{i+1}\}. \end{aligned}$$

We let $ris(\sigma) = |Rise(\sigma)|$ and $des(\sigma) = |Des(\sigma)|$. Then the generating function for the distri-

bution of rises or descents in S_n is given by

$$\sum_{n \geq 0} \frac{t^n}{n!} \sum_{\sigma \in S_n} x^{\text{ris}(\sigma)} = \frac{1-x}{-x + e^{t(x-1)}}, \quad (3)$$

see Stanley [21]. A permutation $\sigma = \sigma_1 \dots \sigma_n \in S_n$ is alternating if

$$\sigma_1 < \sigma_2 > \sigma_3 < \sigma_4 > \sigma_5 < \dots$$

or, equivalently, if $\text{Ris}(\sigma)$ equals the set of odd numbers which are less than n . Let A_n denote the number of alternating permutations of S_n . Then André [1], [2] proved that

$$\sec(t) = 1 + \sum_{n \geq 1} \frac{A_{2n} t^{2n}}{(2n)!} \quad (4)$$

and

$$\tan(t) = \sum_{n \geq 0} \frac{A_{2n+1} t^{2n+1}}{(2n+1)!}. \quad (5)$$

If F is any filling of a $k \times n$ -rectangle with distinct positive integers such that elements in each column increase, reading from bottom to top, then we let $\text{red}(F)$ denote the element of $\mathcal{F}_{n,k}$ which results from F by replacing the i -th smallest element of F by i . For example, Figure 2 demonstrates a filling, F , with its corresponding reduced filling, $\text{red}(F)$.

	12	16	22		7	10	12
	8	15	17		5	9	11
	6	10	13		3	6	8
	1	7	5		1	4	2
F				$\text{red}(F)$			

Figure 2: An example of $F \in \mathcal{F}_{3,4}$ and $\text{red}(F)$.

If $F \in \mathcal{F}_{n,k}$ and $1 \leq c_1 < \dots < c_j \leq n$, then we let $F[c_1, \dots, c_j]$ be the filling of the $k \times j$ rectangle where the elements in column a of $F[c_1, \dots, c_j]$ equal the elements in column c_a in F for $a = 1, \dots, j$. We can then extend the usual pattern matching definitions from permutations to elements of $\mathcal{F}_{n,k}$ as follows.

Definition 1. Let P be an element of $\mathcal{F}_{j,k}$ and $F \in \mathcal{F}_{n,k}$ where $j \leq n$. Then we say

1. P **occurs** in F if there are $1 \leq i_1 < i_2 < \dots < i_j \leq n$ such that $\text{red}(F[i_1, \dots, i_j]) = P$,
2. F **avoids** P if there is no occurrence of P in F , and
3. there is a **P -match in F starting at position i** if $\text{red}(F[i, i+1, \dots, i+j-1]) = P$.

Again, when $k = 1$, then $\mathcal{F}_{n,1} = S_n$, where S_n is the symmetric group, and our definitions reduce to the standard definitions that have appeared in the pattern matching literature. We note that Kitaev, Mansour, and Vella [11] have studied pattern matching in matrices which is a more general setting than the one we are considering in this paper.

$$P = \begin{bmatrix} 3 & 6 & 9 \\ 2 & 5 & 8 \\ 1 & 4 & 7 \end{bmatrix}$$
$$G = \begin{array}{|c|c|c|c|c|c|} \hline 4 & 7 & 11 & 16 & 18 & 14 \\ \hline 2 & 6 & 10 & 13 & 17 & 9 \\ \hline 1 & 5 & 8 & 12 & 15 & 3 \\ \hline \end{array}$$

One can easily extend these notions to sets of elements of $\mathcal{F}_{j,k}$. That is, suppose that $\Upsilon \subseteq \mathcal{F}_{j,k}$. Then $F \in \mathcal{F}_{n,k}$ has an Υ -match at place i provided $\text{red}(F[i, i+1, \dots, i+j-1]) \in \Upsilon$. Let $\Upsilon\text{-mch}(F)$ and $\Upsilon\text{-nlap}(F)$ be the number of Υ -matches and nonoverlapping Υ -matches in F , respectively.

$P_1^{(2,2)}$

2	4
1	3

$P_2^{(2,2)}$

3	4
1	2

If $P \in \mathcal{F}_{2,k}$, then we define $Full_n^P$ to be the set of $F \in \mathcal{F}_{n,k}$ with $P\text{-mch}(F) = n - 1$, i.e. the set of $F \in \mathcal{F}_{n,k}$ with the property that there are P -matches in F starting at positions $1, 2, \dots, n - 1$. We let $full_n^P = |Full_n^P|$, and by convention, define $full_1^P = 1$. For example, if P is the element of $\mathcal{F}_{2,k}$ that has the elements $1, \dots, k$ in the first column and the elements $k + 1, \dots, 2k$ in the second column, then it is easy to see that $full_n^P = 1$ for all $n \geq 1$ since the only element of $F \in \mathcal{F}_{n,k}$ with $P\text{-mch}(F) = n - 1$ has the entries $(i - 1)k + 1, \dots, (i - 1)k + k$ in the i -th column for $i = 1, \dots, n$. More generally, if Υ is a subset of $\mathcal{F}_{2,k}$, then we define $Full_n^\Upsilon$ to be the set of $F \in \mathcal{F}_{n,k}$ such that $\Upsilon\text{-mch}(F) = n - 1$ and let $full_n^\Upsilon = |Full_n^\Upsilon|$. Again, we use

the convention that $full_1^\Upsilon = 1$. For example, if $\Upsilon = St_{2^k}$ is the set of all standard tableaux of shape 2^k , then it easy to see that a $F \in \mathcal{F}_{n,k}$ has $\Upsilon\text{-mch}(F) = n - 1$ if and only if F is standard tableaux of shape n^k , so

$$full_n^\Upsilon = st_{n^k} = \frac{(kn)!}{\prod_{i=0}^{k-1} (i+n) \downarrow_n}.$$

If Υ is a subset of $\mathcal{F}_{2,k}$, then we shall be interested in the following three generating functions:

$$\begin{aligned} D^\Upsilon(x, t) &= 1 + \sum_{n \geq 1} \frac{t^n}{(kn)!} \sum_{F \in \mathcal{F}_{n,k}} x^{\Upsilon\text{-mch}(F)}, \\ A^\Upsilon(t) &= D^\Upsilon(0, t) = 1 + \sum_{n \geq 1} \frac{A_{n,k}^\Upsilon t^n}{(kn)!}, \text{ and} \\ N^\Upsilon(x, t) &= 1 + \sum_{n \geq 1} \frac{t^n}{(kn)!} \sum_{F \in \mathcal{F}_{n,k}} x^{\Upsilon\text{-nlap}(F)} \end{aligned}$$

where $A_{n,k}^\Upsilon$ is equal to the number of $F \in \mathcal{F}_{n,k}$ that have no Υ -matches. If Υ consists of a single element P , then we write $D^P(x, t)$, $A^P(t)$, and $N^P(x, t)$ for $D^\Upsilon(x, t)$, $A^\Upsilon(t)$, and $N^\Upsilon(x, t)$, respectively. Clearly, the generating functions $D^\Upsilon(x, t)$ where $\Upsilon \subseteq St_{n^k}$ can be viewed as analogues of the generating function for rises as described in (3).

We shall prove the following general theorems concerning the generating functions $D^\Upsilon(x, t)$, $A^\Upsilon(t)$, and $N^\Upsilon(x, t)$.

Theorem 2. *For all $\Upsilon \subseteq \mathcal{F}_{2,k}$,*

$$D^\Upsilon(x, t) = \frac{1 - x}{1 - x + \sum_{n \geq 1} \frac{((x-1)t)^n}{(kn)!} full_n^\Upsilon}. \quad (6)$$

Theorem 3. *For all $\Upsilon \subseteq \mathcal{F}_{2,k}$,*

$$N^\Upsilon(x, t) = \frac{A^\Upsilon(t)}{1 - x(1 + (\frac{t}{k!} - 1)A^\Upsilon(t))}. \quad (7)$$

Theorem 2 is proved by applying a ring homomorphism defined on the ring, Λ , of symmetric functions over infinitely many variables, x_1, x_2, \dots , to a simple symmetric function identity. There has been a long line of research, [3], [4], [5], [12], [13], [15], [16], [17], [19], [22], that shows that a large number of generating functions for permutation statistics can be obtained by applying homomorphisms defined on Λ to simple symmetric function identities. Theorem 3 is an analogue of a result of Kitaev [10].

By our remarks above, we have the following corollaries.

Corollary 4. *Let St_{2^k} denote the set of standard tableaux of shape 2^k , then*

$$D^{St_{2^k}}(x, t) = \frac{1 - x}{1 - x + (x - 1) \frac{t}{k!} + \sum_{n \geq 2} \frac{((x-1)t)^n}{\prod_{i=1}^k (n+i-1) \downarrow_n}} \quad (8)$$

and

$$A^{St_{2^k}}(t) = \frac{1}{1 - \frac{t}{k!} + \sum_{n \geq 2} \frac{(-t)^n}{\prod_{i=1}^k (n+i-1) \downarrow_n}}. \quad (9)$$

Corollary 5. Let P be the element of St_{2k} that has $1, \dots, k$ in the first column, then

$$D^P(x, t) = \frac{1-x}{1-x + (x-1)\frac{t}{k!} + \sum_{n \geq 2} \frac{((x-1)t)^n}{(kn)!}} \quad (10)$$

and

$$A^P(t) = \frac{1}{1 - \frac{t}{k!} + \sum_{n \geq 2} \frac{(-t)^n}{(kn)!}}. \quad (11)$$

We have developed similar results for other 2-column patterns; the key is to be able to compute $full_n^P$. For example, we can prove the following.

Theorem 6. 1. If $P_2^{(2,2)} = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$, then $full_n^{P_2^{(2,2)}} = C_{n-1}$ where $C_n = \frac{1}{n+1} \binom{2n}{n}$ is the n -th Catalan number.

2. If $P_1^{(2,2,2)} = \begin{bmatrix} 4 & 6 \\ 2 & 5 \\ 1 & 3 \end{bmatrix}$, then $full_n^{P_1^{(2,2,2)}} = \frac{1}{2n+1} \binom{3n}{n}$.

Let \mathbb{P} denote the set of positive integers $\{1, 2, \dots\}$, $\mathbb{O} = \{1, 3, 5, 7, \dots\}$ denote the set of odd numbers in \mathbb{P} , and $\mathbb{E} = \{2, 4, 6, 8, \dots\}$ denote the set of even numbers in \mathbb{P} . For $n \in \mathbb{P}$, let $[n] = \{1, \dots, n\}$, $\mathbb{O}_n = [n] \cap \mathbb{O}$, and $\mathbb{E}_n = [n] \cap \mathbb{E}$. Now suppose that Υ is a subset of $\mathcal{F}_{2,k}$ and $F \in \mathcal{F}_{n,k}$. Then we let

1. $S^\Upsilon(F)$ denote the set of i such that there is an Υ -match of F starting at position i ,
2. $\mathcal{F}_{n,k}^{(2),\Upsilon}$ denote the set of $F \in \mathcal{F}_{n,k}$ such that $\mathbb{O}_n \subseteq S^\Upsilon(F)$, and
3. $\Upsilon\text{-mch}^{(2)}(F)$ denote the number of i such that $2i \in S^\Upsilon(F)$.

We say that $F \in \mathcal{F}_{n,k}$ is **Υ -alternating** if $S^\Upsilon(F) = \mathbb{O}_n$. That is, $F \in \mathcal{F}_{n,k}$ is Υ -alternating if $\text{red}(F[2i+1, 2i+2]) \in \Upsilon$ for all $0 \leq i \leq (n-2)/2$ and $\text{red}(F[2i, 2i+1]) \notin \Upsilon$ for all $0 < i \leq (n-1)/2$. We let Alt_n^Υ denote the number of Υ -alternating elements of $\mathcal{F}_{n,k}$.

We shall prove the following general theorems about Υ -alternating fillings.

Theorem 7. Let Υ be a subset of $\mathcal{F}_{2,k}$. Then

$$1 + \sum_{n \geq 1} \frac{t^{2n}}{(2kn)!} \sum_{F \in \mathcal{F}_{2n,k}^{(2)}} x^{\Upsilon\text{-mch}^{(2)}(F)} = \frac{1}{1 - \sum_{n \geq 1} \frac{(x-1)^{n-1} t^{2n}}{(2kn)!} full_{2n}^\Upsilon}.$$

Theorem 8. Let Υ be a subset of $\mathcal{F}_{2,k}$. Then

$$\sum_{n \geq 1} \frac{t^{2n-1}}{(k(2n-1))!} \sum_{F \in \mathcal{F}_{2n-1,k}^{(2)}} x^{\Upsilon\text{-mch}^{(2)}(F)} = \frac{\sum_{n \geq 1} \frac{(x-1)^{n-1} t^{2n-1}}{(k(2n-1))!} full_{2n-1}^\Upsilon}{1 - \sum_{n \geq 1} \frac{(x-1)^{n-1} t^{2n}}{(2kn)!} full_{2n}^\Upsilon}.$$

Theorems 7 and 8 are also proved by applying a ring homomorphism defined on Λ to simple symmetric function identities. Putting $x = 0$ in Theorems 7 and 8, we obtain the following generating functions for Υ -alternating elements of $\mathcal{F}_{n,k}$ which can be viewed as generalizations of (4) and (5).

Corollary 9.

$$1 + \sum_{n \geq 1} \frac{Alt_{2n}^{\Upsilon} t^{2n}}{(2kn)!} = \frac{1}{1 + \sum_{n \geq 1} \frac{(-1)^n t^{2n}}{(2kn)!} full_{2n}^{\Upsilon}} \text{ and} \quad (12)$$

$$\sum_{n \geq 1} \frac{Alt_{2n-1}^{\Upsilon} t^{2n-1}}{(k(2n-1))!} = \frac{\sum_{n \geq 1} \frac{(-1)^{n-1} t^{2n-1}}{(k(2n-1))!} full_{2n-1}^{\Upsilon}}{1 + \sum_{n \geq 1} \frac{(-1)^n t^{2n}}{(2kn)!} full_{2n}^{\Upsilon}}. \quad (13)$$

The outline of this paper is the following. In Section 2, we shall review the necessary background on symmetric functions needed to prove Theorems 2, 7, and 8. Then in section 3, we shall give the proofs of Theorems 2, 7, and 8. In section 4, we shall give the proof of Theorem 3. In Section 5, we shall find formulas for $full_n^{\Upsilon}$ where Υ are subsets of the standard tableaux of shape 2^2 and we shall completely classify all standard tableaux, $P \in St_{2k}$, such that $full_n^P = 1$ for all $n \geq 1$. In addition, we shall briefly outline some methods that we have used to compute $full_n^P$ for certain standard tableaux $P \in St_{2k}$ where $k \geq 3$.

2 Symmetric Functions

In this section we give the necessary background on symmetric functions needed for our proofs of Theorems 2, 7, and 8. We shall consider the ring of symmetric functions, Λ , over infinitely many variables x_1, x_2, \dots . The homogeneous symmetric functions, $h_n \in \Lambda$, and elementary symmetric functions, $e_n \in \Lambda$, are defined by the generating functions

$$H(t) = \sum_{n \geq 0} h_n t^n = \prod_{i=1}^{\infty} \frac{1}{1 - x_i t} \text{ and } E(t) = \sum_{n \geq 0} e_n t^n = \prod_{i=1}^{\infty} (1 + x_i t).$$

The n -th power symmetric function, $p_n \in \Lambda$, is defined as

$$p_n = \sum_{i=1}^{\infty} x_i^n.$$

Let $\lambda = (\lambda_1, \dots, \lambda_\ell)$ be an integer partition; that is, λ is a finite sequence of weakly increasing non-negative integers. Let $\ell(\lambda)$ denote the number of nonzero integers in λ . If the sum of these integers is n , we say that λ is a partition of n and write $\lambda \vdash n$. For any partition $\lambda = (\lambda_1, \dots, \lambda_\ell)$, define $h_\lambda = h_{\lambda_1} \cdots h_{\lambda_\ell}$, $e_\lambda = e_{\lambda_1} \cdots e_{\lambda_\ell}$, and $p_\lambda = p_{\lambda_1} \cdots p_{\lambda_\ell}$. The well-known fundamental theorem of symmetric functions, see [14], says that $\{e_\lambda : \lambda \vdash n\}$ is a basis for Λ_n , the space of symmetric functions which are homogeneous of degree n . Equivalently, the fundamental theorem of symmetric functions states that $\{e_0, e_1, \dots\}$ is an algebraically independent set of generators for the ring Λ . It follows that one can completely specify a ring homomorphism $\Gamma : \Lambda \rightarrow R$ from Λ into a ring R by giving the values of $\Gamma(e_n)$ for $n \geq 0$.

Next we give combinatorial interpretations to the expansion of h_μ in terms of the elementary symmetric functions. Given partitions $\lambda = (\lambda_1, \dots, \lambda_\ell) \vdash n$ and $\mu \vdash n$, a λ -brick tabloid of shape μ is a filling of the Ferrers diagram of shape μ with bricks of size $\lambda_1, \dots, \lambda_\ell$ such that each brick lies in single row and no two bricks overlap. For example, Figure 5 shows all the λ -brick tabloids of shape μ where $\lambda = (1, 1, 2, 2)$ and $\mu = (4, 2)$.

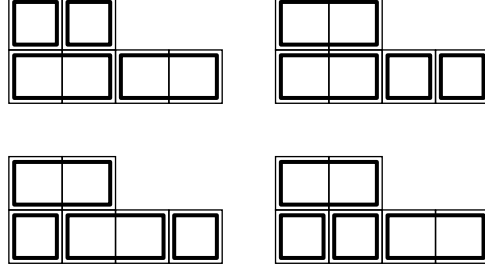


Figure 5: The four $(1, 1, 2, 2)$ -brick tabloids of shape $(4, 2)$.

Let $\mathcal{B}_{\lambda, \mu}$ denote the set of all λ -brick tabloids of shape μ and let $B_{\lambda, \mu} = |\mathcal{B}_{\lambda, \mu}|$. Egecioglu and Remmel proved in [8] that

$$h_{\mu} = \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} B_{\lambda, \mu} e_{\lambda}. \quad (14)$$

If T is a brick tabloid of shape (n) such that the lengths of the bricks, reading from left to right, are b_1, \dots, b_{ℓ} , then we shall write $T = (b_1, \dots, b_{\ell})$. For example, the brick tabloid $T = (2, 3, 1, 4, 2)$ is pictured in Figure 6.



Figure 6: The brick tabloid $T = (2, 3, 1, 4, 2)$.

Next we define a symmetric function $p_{n, \nu}$ whose relationship with e_{λ} , which is very similar to the relationship between h_n and e_{λ} , was first introduced in [13] and [16]. Let ν be a function which maps the set of non-negative integers into a field F . Recursively define $p_{n, \nu} \in \Lambda_n$ by setting $p_{0, \nu} = 1$ and letting

$$p_{n, \nu} = (-1)^{n-1} \nu(n) e_n + \sum_{k=1}^{n-1} (-1)^{k-1} e_k p_{n-k, \nu}$$

for all $n \geq 1$. By multiplying series, this means that

$$\left(\sum_{n \geq 0} (-1)^n e_n t^n \right) \left(\sum_{n \geq 1} p_{n, \nu} t^n \right) = \sum_{n \geq 1} \left(\sum_{k=0}^{n-1} p_{n-k, \nu} (-1)^k e_k \right) t^n = \sum_{n \geq 1} (-1)^{n-1} \nu(n) e_n t^n,$$

where the last equality follows from the definition of $p_{n, \nu}$. Therefore,

$$\sum_{n \geq 1} p_{n, \nu} t^n = \frac{\sum_{n \geq 1} (-1)^{n-1} \nu(n) e_n t^n}{\sum_{n \geq 0} (-1)^n e_n t^n} \quad (15)$$

or, equivalently,

$$1 + \sum_{n \geq 1} p_{n, \nu} t^n = \frac{1 + \sum_{n \geq 1} (-1)^n (e_n - \nu(n) e_n) t^n}{\sum_{n \geq 0} (-1)^n e_n t^n}. \quad (16)$$

If $\nu(n) = 1$ for all $n \geq 1$, then (16) becomes

$$1 + \sum_{n \geq 1} p_{n,1} t^n = \frac{1}{\sum_{n \geq 0} (-1)^n e_n t^n} = 1 + \sum_{n \geq 1} h_n t^n$$

which implies $p_{n,1} = h_n$. Other special cases for ν give well-known generating functions. For example, if $\nu(n) = n$ for $n \geq 1$, then $p_{n,\nu}$ is the power symmetric function p_n . By taking $\nu(n) = (-1)^k \chi(n \geq k+1)$ for some $k \geq 1$, $p_{n,(-1)^k \chi(n \geq k+1)}$ is the Schur function corresponding to the partition $(1^k, n)$. Here for any statement A , we let $\chi(A) = 1$ if A is true and $\chi(A) = 0$ if A is false.

This definition of $p_{n,\nu}$ is desirable because of its expansion in terms of elementary symmetric functions. The coefficient of e_λ in $p_{n,\nu}$ has a nice combinatorial interpretation similar to that of the homogeneous symmetric functions. Suppose T is a brick tabloid of shape (n) and type λ and that the final brick in T has length ℓ . Define the weight of a brick tabloid $w_\nu(T)$ to be $\nu(\ell)$ and let

$$w_\nu(B_{\lambda,(n)}) = \sum_{\substack{T \text{ is a brick tabloid} \\ \text{of shape } (n) \text{ and type } \lambda}} w_\nu(T).$$

When $\nu(n) = 1$ for $n \geq 1$, $B_{\lambda,(n)}$ and $w_\nu(B_{\lambda,(n)})$ are the same. Then Mendes and Remmel [16] proved that

$$p_{n,\nu} = \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} w_\nu(B_{\lambda,(n)}) e_\lambda.$$

3 The proofs of Theorems 2, 7, and 8

In this section, we shall give the proofs of Theorems 2, 7, and 8. We start with the proof of Theorem 2.

Proof of Theorem 2.

Our goal is to prove that for all $\Upsilon \subseteq \mathcal{F}_{2,k}$,

$$D^\Upsilon(x, t) = 1 + \sum_{n \geq 1} \frac{t^n}{(kn)!} \sum_{F \in \mathcal{F}_{n,k}} x^{\Upsilon\text{-mch}(F)} = \frac{1 - x}{1 - x + \sum_{n \geq 1} \frac{((x-1)t)^n}{(kn)!} full_n^\Upsilon}. \quad (17)$$

Define a ring homomorphism $\Gamma : \Lambda \rightarrow \mathbb{Q}[x]$, where $\mathbb{Q}[x]$ is the polynomial ring over the rationals, by setting $\Gamma(e_0) = 1$ and

$$\Gamma(e_n) = \frac{(-1)^{n-1}}{(kn)!} full_n^\Upsilon (x-1)^{n-1} \quad (18)$$

for $n \geq 1$. Then we claim that

$$(kn)! \Gamma(h_n) = \sum_{F \in \mathcal{F}_{n,k}} x^{\Upsilon\text{-mch}(F)} \quad (19)$$

for all $n \geq 1$. That is,

$$\begin{aligned}
(kn)!\Gamma(h_n) &= (kn)! \sum_{\mu \vdash n} (-1)^{n-\ell(\mu)} B_{\mu,(n)} \Gamma(e_\mu) \\
&= (kn)! \sum_{\mu \vdash n} (-1)^{n-\ell(\mu)} \sum_{(b_1, \dots, b_{\ell(\mu)}) \in \mathcal{B}_{\mu,(n)}} \prod_{j=1}^{\ell(\mu)} \frac{(-1)^{b_j-1}}{(kb_j)!} full_{b_j}^\Upsilon(x-1)^{b_j-1} \\
&= \sum_{\mu \vdash n} \sum_{(b_1, \dots, b_{\ell(\mu)}) \in \mathcal{B}_{\mu,(n)}} \binom{kn}{kb_1, \dots, kb_{\ell(\mu)}} \prod_{j=1}^{\ell(\mu)} full_{b_j}^\Upsilon(x-1)^{b_j-1}. \tag{20}
\end{aligned}$$

Next we want to give a combinatorial interpretation to (20). First we interpret $\binom{kn}{kb_1, \dots, kb_{\ell(\mu)}}$ as the number of ways to pick a set partition of $\{1, \dots, kn\}$ into sets $S_1, \dots, S_{\ell(\mu)}$ where $|S_j| = kb_j$ for $j = 1, \dots, \ell(\mu)$. For each set S_j , we interpret $full_{b_j}^\Upsilon$ as the number of ways to arrange the numbers in S_j into a $k \times b_j$ array such that there is an Υ -match starting at positions $1, \dots, b_j - 1$. Finally, we interpret $\prod_{j=1}^{\ell(\mu)} (x-1)^{b_j-1}$ as all ways of picking labels for all the columns of each brick, except the final column, with either an x or a -1 . For completeness, we label the final column of each brick with 1. We shall call all such objects created in this way *filled labeled brick tabloids* and let $\mathcal{H}_{n,k}^\Upsilon$ denote the set of all filled labeled brick tabloids that arise in this way. Thus a $C \in \mathcal{H}_{n,k}^\Upsilon$ consists of a brick tabloid, T , a filling, $F \in \mathcal{F}_{n,k}$, and a labeling, L , of the columns of T with elements from $\{x, 1, -1\}$ such that

1. F has an Υ -match starting at each column that is not a final column a brick,
2. the final column of each brick is labeled with 1, and
3. each column that is not a final column of a brick is labeled with x or -1 .

We then define the weight of C , $w(C)$, to be the product of all the x labels in L and the sign of C , $sgn(C)$, to be the product of all the -1 labels in L . For example, if $n = 12$, $k = 2$, $\Upsilon = St_{2,2}$, and $T = (4, 3, 3, 2)$, then Figure 7 demonstrates such a composite object $C \in \mathcal{H}_{12,2}$ where, $w(C) = x^5$ and $sgn(C) = -1$.

Thus

$$(kn)!\Gamma(h_n) = \sum_{C \in \mathcal{H}_{n,k}^\Upsilon} sgn(C)w(C). \tag{21}$$

x	-1	x	1	-1	x	1	x	x	1	-1	1
5	13	17	21	4	12	15	9	20	23	8	24
3	10	14	16	1	7	11	6	18	19	2	22

Figure 7: A composite object $C \in \mathcal{H}_{12,2}^{St_{2,2}}$.

Next we define a weight-preserving sign-reversing involution $I : \mathcal{H}_{n,k}^\Upsilon \rightarrow \mathcal{H}_{n,k}^\Upsilon$. To define $I(C)$, we scan the columns of $C = (T, F, L)$ from left to right looking for the leftmost column, t , such that either (i) t is labeled with -1 or (ii) t is at the end of a brick, b_j , and the brick immediately following b_j , namely b_{j+1} , has the property that F has an Υ -match starting in each

column of b_j and b_{j+1} except, possibly, the last cell of b_{j+1} . In case (i), $I(C) = (T', F', L')$ where T' is the result of replacing the brick b in T containing t by two bricks, b^* and b^{**} , where b^* contains the t -th column plus all the columns in b to the left of t and b^{**} contains all the columns of b to the right of t , $F' = F$, and L' is the labeling that results from L by changing the label of column t from -1 to 1 . In case (ii), $I(C) = (T', F', L')$ where T' is the result of replacing the bricks b_j and b_{j+1} in T by a single brick b , $F' = F$, and L' is the labeling that results from L by changing the label of column t from 1 to -1 . If neither case (i) or case (ii) applies, then we let $I(C) = C$. For example, if C is the element of $\mathcal{H}_{12,2}^{St_2^2}$ pictured in Figure 7, then $I(C)$ is pictured in Figure 8.

x	1	x	1	-1	x	1	x	x	1	-1	1
5	13	17	21	4	12	15	9	20	23	8	24
3	10	14	16	1	7	11	6	18	19	2	22

Figure 8: $I(C)$ for C in Figure 7.

It is easy to see that $I^2(C) = C$ for all $C \in \mathcal{H}_{n,k}^\Upsilon$ and that if $I(C) \neq C$, then $\text{sgn}(C)w(C) = -\text{sgn}(I(C))w(I(C))$. Hence I is a weight-preserving and sign-reversing involution that shows

$$(kn)!\Gamma(h_n) = \sum_{C \in \mathcal{H}_{n,k}^\Upsilon, I(C)=C} \text{sgn}(C)w(C). \quad (22)$$

Thus, we must examine the fixed points, $C = (T, F, L)$, of I . First, there can be no -1 labels in L so that $\text{sgn}(C) = 1$. Moreover, if b_j and b_{j+1} are two consecutive bricks in T and t is the last column of b_j , then it cannot be the case that there is an Υ -match starting at position t in F since otherwise we could combine b_j and b_{j+1} . It follows that $\text{sgn}(C)w(C) = x^{\Upsilon\text{-mch}(F)}$. For example, Figure 9 shows a fixed point of I in the case $n = 12$, $k = 2$, and $\Upsilon = St_2^2$.

Vice versa, if $F \in \mathcal{F}_{n,k}$, then we can create a fixed point, $C = (T, F, L)$, by forming bricks in T that end at columns that are not the start of an Υ -match in F , labeling each column that is the start of an Υ -match in F with x , and labeling the remaining columns with 1 . Thus we have shown that

$$(kn)!\Gamma(h_n) = \sum_{F \in \mathcal{F}_{n,k}} x^{\Upsilon\text{-mch}(F)}$$

as desired.

x	x	x	1	x	x	1	x	x	1	x	1
5	13	17	21	4	12	15	9	20	23	8	24
3	10	14	16	1	7	11	6	18	19	2	22

Figure 9: A fixed point of I .

Applying Γ to the identity $H(t) = \frac{1}{E(-t)}$, we get

$$\begin{aligned}
\sum_{n \geq 0} \Gamma(h_n) t^n &= \sum_{n \geq 0} \frac{t^n}{(kn)!} \sum_{F \in \mathcal{F}_{n,k}} x^{\Upsilon\text{-mch}(F)} \\
&= \frac{1}{1 + \sum_{n \geq 1} (-t)^n \Gamma(e_n)} \\
&= \frac{1}{1 + \sum_{n \geq 1} (-1)^n t^n \frac{(-1)^{n-1} (x-1)^{n-1}}{(kn)!} full_n^\Upsilon} \\
&= \frac{1-x}{1-x + \sum_{n \geq 1} \frac{((x-1)t)^n}{(kn)!} full_n^\Upsilon}
\end{aligned}$$

which proves (17).

Putting $x = 0$ in (17) immediately yields the following corollary.

Corollary 10. *Let $\Upsilon \subseteq \mathcal{F}_{2,k}$ and $A_{n,k}^\Upsilon$ be the number of $F \in \mathcal{F}_{n,k}$ that have no Υ -matches. Then*

$$A^\Upsilon(t) = 1 + \sum_{n \geq 1} \frac{A_{n,k}^\Upsilon t^n}{(kn)!} = \frac{1}{1 + \sum_{n \geq 1} \frac{(-t)^n}{(kn)!} full_n^\Upsilon}. \quad (23)$$

Recall that if $\Upsilon \subseteq \mathcal{F}_{k,2}$, then $\mathcal{F}_{n,k}^{(2),\Upsilon}$ denotes the set of $F \in \mathcal{F}_{n,k}$ that have Υ -matches starting at all the odd-numbered columns and that for $F \in \mathcal{F}_{n,k}^{(2),\Upsilon}$, $\Upsilon\text{-mch}^{(2)}(F)$ denotes the number of i such that F has an Υ -match starting at position $2i$.

Next we give the proof of Theorem 7.

Proof of Theorem 7.

Our goal is to prove that if Υ is a subset of $\mathcal{F}_{2,k}$, then

$$\begin{aligned}
D^{(2)}(x, t) &= 1 + \sum_{n \geq 1} \frac{t^{2n}}{(2kn)!} \sum_{F \in \mathcal{F}_{2n,k}^{(2)}} x^{\Upsilon\text{-mch}^{(2)}(F)} \\
&= \frac{1}{1 - \sum_{n \geq 1} \frac{(x-1)^{n-1} t^{2n}}{(2kn)!} full_{2n}^\Upsilon}. \quad (24)
\end{aligned}$$

The proof of Theorem 7 is very similar to the proof of Theorem 2. We only have to modify our ring homomorphism slightly. That is, define a ring homomorphism $\Gamma^{(2)} : \Lambda \rightarrow \mathbb{Q}[x]$ by setting $\Gamma^{(2)}(e_0) = 1$, $\Gamma^{(2)}(e_{2n+1}) = 0$ for all $n \geq 0$, and

$$\Gamma(e_{2n}) = \frac{(-1)^{2n-1}}{(2kn)!} full_{2n}^\Upsilon (x-1)^{n-1} \quad (25)$$

for $n \geq 1$.

Then we claim that for $n \geq 1$,

$$\Gamma^{(2)}(h_{2n-1}) = 0 \quad (26)$$

and

$$(2kn)!\Gamma^{(2)}(h_{2n}) = \sum_{F \in \mathcal{F}_{2n,k}^{(2)}} x^{\Upsilon\text{-mch}^{(2)}(F)}. \quad (27)$$

Note that

$$\Gamma^{(2)}(h_n) = \sum_{\mu \vdash n} (-1)^{n-\ell(\mu)} B_{\mu,(n)} \Gamma^{(2)}(e_\mu). \quad (28)$$

Now if n is odd, then every $\mu \vdash n$ must have an odd part and hence our definitions force $\Gamma^{(2)}(e_\mu) = 0$. Thus $\Gamma^{(2)}(h_{2n-1}) = 0$ for all $n \geq 1$. On the other hand, if n is even, then we can restrict ourselves to those partitions in (28) that contain only even parts. For any partition $\lambda = (\lambda_1, \dots, \lambda_\ell)$ of n , we let 2λ denote the partition $(2\lambda_1, \dots, 2\lambda_\ell)$. Then for $n \geq 1$ we have

$$\begin{aligned} (2kn)!\Gamma^{(2)}(h_{2n}) &= (2kn)! \sum_{\mu \vdash n} (-1)^{2n-\ell(\mu)} B_{2\mu,(2n)} \Gamma^{(2)}(e_{2\mu}) \\ &= (2kn)! \sum_{\mu \vdash n} (-1)^{2n-\ell(\mu)} \sum_{(2b_1, \dots, 2b_{\ell(\mu)}) \in \mathcal{B}_{2\mu,(2n)}} \prod_{j=1}^{\ell(\mu)} \frac{(-1)^{2b_j-1}}{(2kb_j)!} full_{2b_j}^{\Upsilon}(x-1)^{b_j-1} \\ &= \sum_{\mu \vdash n} \sum_{(2b_1, \dots, 2b_{\ell(\mu)}) \in \mathcal{B}_{2\mu,(2n)}} \binom{2kn}{2kb_1, \dots, 2kb_{\ell(\mu)}} \prod_{j=1}^{\ell(\mu)} full_{2b_j}^{\Upsilon}(x-1)^{b_j-1}. \end{aligned} \quad (29)$$

Again we can give a combinatorial interpretation to (29). First we interpret $\binom{2kn}{2kb_1, \dots, 2kb_{\ell(\mu)}}$ as the number of ways to pick a set partition of $\{1, \dots, 2kn\}$ into sets $S_1, \dots, S_{\ell(\mu)}$ where $|S_j| = 2kb_j$ for $j = 1, \dots, \ell(\mu)$. For each set S_j , we interpret $full_{2b_j}^{\Upsilon}$ as the number of ways to arrange the numbers in S_j into a $k \times 2b_j$ array such that there is an Υ -match starting at positions $1, \dots, 2b_j-1$ in the brick $2b_j$. Finally, we interpret $\prod_{j=1}^{\ell(\mu)} (x-1)^{b_j-1}$ as all ways of picking labels for all the even numbered columns of each brick, except the final even numbered column in the brick, with either an x or a -1 . For completeness, we label the final even numbered column of each brick with 1. We shall call all such objects created in this way *filled labeled brick tabloids* and let $\mathcal{H}_{2n,k}^{(2),\Upsilon}$ denote the set of all filled labeled brick tabloids that arise in this way. Thus a $C \in \mathcal{H}_{2n,k}^{(2),\Upsilon}$ consists of a brick tabloid, T , a filling, $F \in \mathcal{F}_{2n,k}$, and a labeling, L , of the even numbered columns of T with elements from $\{x, 1, -1\}$ such that

1. all bricks have even size,
2. F has an Υ -match starting at each column that is not a final column of its brick,
3. the final column of each brick is labeled with 1, and
4. each even numbered column that is not a final column of a brick is labeled with x or -1 .

We then define the weight of C , $w(C)$, to be the product of all the x labels in L and the sign of C , $sgn(C)$, to be the product of all the -1 labels in L . For example, if $n = 12$, $k = 2$, $\Upsilon = St_{2,2}$, and $T = (4, 2, 4, 2)$, then Figure 10 pictures such a filled labeled brick tabloid $C \in \mathcal{H}_{12,2}^{(2),\Upsilon}$ where $w(C) = x$ and $sgn(C) = -1$.

Thus

$$(2kn)!\Gamma^{(2)}(h_{2n}) = \sum_{C \in \mathcal{H}_{2n,k}^{(2),\Upsilon}} \text{sgn}(C)w(C). \quad (30)$$

1		-1		1		x		1		1	
5	13	12	16	17	21	11	15	20	23	8	24
3	10	1	4	7	14	6	9	18	19	2	22

Figure 10: A $C \in \mathcal{H}_{12,2}^{(2),St_2}$.

Next we define a weight-preserving sign-reversing involution $I : \mathcal{H}_{2n,k}^{(2),\Upsilon} \rightarrow \mathcal{H}_{2n,k}^{(2),\Upsilon}$ essentially in the same way as in Theorem 2. That is, to define $I(C)$, we scan the cells of $C = (T, F, L)$ from left to right looking for the leftmost column, t , such that either (i) t is labeled with -1 or (ii) t is at the end of a brick, $2b_j$, and the brick immediately following $2b_j$, namely $2b_{j+1}$, has the property that F has a Υ -match starting in each column of $2b_j$ and $2b_{j+1}$ except, possibly, the last cell of $2b_{j+1}$. In case (i), $I(C) = (T', F', L')$ where T' is the result of replacing the brick $2b$ in T containing t by two bricks, $2b^*$ and $2b^{**}$, where $2b^*$ contains the t -th column plus all the cells in $2b$ to the left of t and $2b^{**}$ contains all the columns of $2b$ to the right of t , $F' = F$, and L' is the labeling that results from L by changing the label of column t from -1 to 1 . In case (ii), $I(C) = (T', F', L')$ where T' is the result of replacing the bricks $2b_j$ and $2b_{j+1}$ in T by a single brick $2b$, $F' = F$, and L' is the labeling that results from L by changing the label of column t from 1 to -1 . If neither case (i) or case (ii) applies, then we let $I(C) = C$. For example, if C is the element of $\mathcal{H}_{12,2}^{(2),St_2}$ pictured in Figure 10, then $I(C)$ is pictured in Figure 11.

1		1		1		x		1		1	
5	13	12	16	17	21	11	15	20	23	8	24
3	10	1	4	7	14	6	9	18	19	2	22

Figure 11: $I(C)$ for C in Figure 10.

It is easy to see that the involution I is weight-preserving and sign-reversing and hence shows that

$$(2kn)!\Gamma^{(2)}(h_{2n}) = \sum_{C \in \mathcal{H}_{n,k}^{(2),\Upsilon}, I(C)=C} \text{sgn}(C)w(C). \quad (31)$$

Thus, we must examine the fixed points $C = (T, F, L)$ of I . First, by construction, F must have Υ -matches starting at all odd positions. Hence $F \in \mathcal{F}_{2n,k}^{(2)}$. As before, there can be no -1 labels in L so that $\text{sgn}(C) = 1$. Moreover, if $2b_j$ and $2b_{j+1}$ are two consecutive bricks in T and t is the last column of $2b_j$, then it cannot be the case that there is an Υ -match starting at position t in F since otherwise we could combine $2b_j$ and $2b_{j+1}$. It follows that $\text{sgn}(C)w(C) = x^{\Upsilon\text{-mch}^{(2)}(F)}$.

Vice versa, if $F \in \mathcal{F}_{2n,k}^{(2)}$, then we can create a fixed point, $C = (T, F, L)$, by forming bricks in T that end at even numbered columns that are not the start of an Υ -match in F , and labeling each even numbered column that is the start of an Υ -match in F with x , and labeling the

remaining even numbered columns with 1. Thus we have shown that

$$(2kn)! \Gamma^{(2)}(h_{2n}) = \sum_{F \in \mathcal{F}_{2n,k}^{(2)}} x^{\Upsilon\text{-mch}^{(2)}(F)}$$

as desired.

Applying $\Gamma^{(2)}$ to the identity $H(t) = \frac{1}{E(-t)}$, we get

$$\begin{aligned} \sum_{n \geq 0} \Gamma(h_n) t^n &= \sum_{n \geq 0} \frac{t^{2n}}{(2kn)!} \sum_{F \in \mathcal{F}_{2n,k}^{(2)}} x^{\Upsilon\text{-mch}^{(2)}(F)} \\ &= \frac{1}{1 + \sum_{n \geq 1} (-t)^{2n} \Gamma^{(2)}(e_{2n})} \\ &= \frac{1}{1 + \sum_{n \geq 1} t^{2n} \frac{(-1)^{2n-1} (x-1)^{n-1}}{(2kn)!} full_{2n}^{\Upsilon}} \\ &= \frac{1}{1 - \sum_{n \geq 1} \frac{(x-1)^{n-1} t^{2n}}{(2kn)!} full_{2n}^{\Upsilon}} \end{aligned}$$

which proves (24).

As noted in the introduction, putting $x = 0$ in (24) gives us the generating function for the number of Υ -alternating elements of $\mathcal{F}_{2n,k}$. That is, if Υ is a subset of $\mathcal{F}_{2,k}$, then

$$1 + \sum_{n \geq 1} \frac{Alt_{2n}^{\Upsilon} t^{2n}}{(2kn)!} = \frac{1}{1 + \sum_{n \geq 1} \frac{(-1)^n t^{2n}}{(2kn)!} full_{2n}^{\Upsilon}}. \quad (32)$$

Next we give the proof of Theorem 8.

Proof of Theorem 8.

Our goal is to prove that if Υ is a subset of $\mathcal{F}_{2,k}$, then

$$\begin{aligned} D^{(2)}(x, t) &= \sum_{n \geq 0} \frac{t^{2n+1}}{((2n+1)k)!} \sum_{F \in \mathcal{F}_{2n+1,k}^{(2)}} x^{\Upsilon\text{-mch}^{(2)}(F)} \\ &= \frac{\sum_{n \geq 1} \frac{(x-1)^{n-1} t^{2n-1}}{(k(2n-1))!} full_{2n-1}^{\Upsilon}}{1 - \sum_{n \geq 1} \frac{(x-1)^{n-1} t^{2n}}{(2kn)!} full_{2n}^{\Upsilon}}. \end{aligned} \quad (33)$$

Let $\Gamma^{(2)} : \Lambda \rightarrow \mathbb{Q}[x]$ be the ring homomorphism defined in Theorem 7. Define $\nu : \mathbb{P} \rightarrow \mathbb{Q}$ by setting $\nu(n) = 0$ if n is odd and setting

$$\nu(2n) = \frac{full_{2n-1}^{\Upsilon}(2kn)!}{full_{2n}^{\Upsilon}(k(2n-1))!} \quad (34)$$

for $n \geq 1$. We have defined ν so that for $n \geq 1$,

$$\begin{aligned} \nu(2n) \Gamma^{(2)}(e_{2n}) &= \frac{full_{2n-1}^{\Upsilon}(2kn)!}{full_{2n}^{\Upsilon}(k(2n-1))!} \frac{(-1)^{2n-1} (x-1)^{n-1}}{(2kn)!} full_{2n}^{\Upsilon} \\ &= \frac{(-1)^{2n-1} (x-1)^{n-1}}{(k(2n-1))!} full_{2n-1}^{\Upsilon}. \end{aligned} \quad (35)$$

Then we claim that for $n \geq 0$,

$$\Gamma^{(2)}(p_{2n+1,\nu}) = 0 \quad (36)$$

and

$$(k(2n+1))!\Gamma^{(2)}(p_{2n+2,\nu}) = \sum_{F \in \mathcal{F}_{2n+1,k}^{(2)}} x^{\Upsilon\text{-mch}^{(2)}(F)}. \quad (37)$$

Note that

$$\Gamma^{(2)}(p_{n,\nu}) = \sum_{\mu \vdash n} (-1)^{n-\ell(\mu)} w_\nu(B_{\mu,(n)}) \Gamma^{(2)}(e_\mu). \quad (38)$$

Now if n is odd, then every $\mu \vdash n$ must have an odd part and hence our definitions force $\Gamma^{(2)}(e_\mu) = 0$. Thus $\Gamma^{(2)}(p_{2n+1,\nu}) = 0$ for all $n \geq 0$. On the other hand, if n is even, then we can restrict ourselves to those partitions which contain only even parts in (38). Thus for $n \geq 1$ we have

$$\begin{aligned} & (k(2n+1))!\Gamma^{(2)}(p_{2n+2,\nu}) = \\ & (k(2n+1))! \sum_{\mu \vdash n+1} (-1)^{2n+2-\ell(\mu)} w_\nu(B_{2\mu,(2n+2)}) \Gamma^{(2)}(e_{2\mu}) = \\ & (k(2n+1))! \sum_{\mu \vdash n+1} (-1)^{2n+2-\ell(\mu)} \times \\ & \sum_{(2b_1, \dots, 2b_{\ell(\mu)}) \in \mathcal{B}_{2\mu,(2n+1)}} \frac{full_{2b_{\ell(\mu)}-1}^{\Upsilon}(2kb_{\ell(\mu)})!}{full_{2b_{\ell(\mu)}}^{\Upsilon}(k(2b_{\ell(\mu)}-1))!} \prod_{j=1}^{\ell(\mu)} \frac{(-1)^{2b_j-1}}{(2kb_j)!} full_{2b_j}^{\Upsilon}(x-1)^{b_j-1} = \\ & \sum_{\mu \vdash n+1} \sum_{(2b_1, \dots, 2b_{\ell(\mu)}) \in \mathcal{B}_{2\mu,(2n+1)}} \binom{k(2n+1)}{2kb_1, \dots, 2kb_{\ell(\mu)-1}, k(2b_{\ell(\mu)}-1)} \times \\ & (x-1)^{b_{\ell(\mu)}-1} full_{2b_{\ell(\mu)}-1}^{\Upsilon} \prod_{j=1}^{\ell(\mu)-1} full_{2b_j}^{\Upsilon}(x-1)^{b_j-1}. \end{aligned} \quad (39)$$

As before, we must give a combinatorial interpretation to (39). First we interpret $\binom{k(2n+1)}{2kb_1, \dots, 2kb_{\ell(\mu)-1}, k(2b_{\ell(\mu)}-1)}$ as the number of ways to pick a set partition of $\{1, \dots, k(2n+1)\}$ into sets $S_1, \dots, S_{\ell(\mu)}$ where $|S_j| = 2kb_j$ for $j = 1, \dots, \ell(\mu) - 1$ and $|S_{\ell(\mu)}| = k(2b_{\ell(\mu)} - 1)$. For each set S_j with $j < \ell(\mu)$, we interpret $full_{2b_j}^{\Upsilon}$ as the number of ways to arrange the numbers in S_j into a $k \times 2b_j$ array such that there is an Υ -match starting at positions $1, \dots, 2b_j - 1$. We interpret $full_{2b_{\ell(\mu)}-1}^{\Upsilon}$ as arranging the numbers in $S_{\ell(\mu)}$ in a $k \times (2b_{\ell(\mu)} - 1)$ array such that there is an Υ -match starting at positions $1, \dots, 2b_{\ell(\mu)} - 2$. In this case, we imagine that the array fills the first $2b_{\ell(\mu)} - 1$ columns of the brick $2b_{\ell(\mu)}$ and the last column is left blank. Finally, we interpret $\prod_{j=1}^{\ell(\mu)} (x-1)^{b_j-1}$ as all ways of picking labels of all the even numbered columns of each brick, except the final even numbered column in the brick, with either an x or a -1 . For completeness, we label the final even numbered column in each brick with 1. We call all such objects created in this way *filled labeled brick tabloids* and let $\mathcal{K}_{2n+2,k}^{(2),\Upsilon}$ denote the set of all filled labeled brick tabloids that arise in this way. Thus, a $C \in \mathcal{K}_{2n+2,k}^{(2),\Upsilon}$ consists of a brick tabloid, T , a filling, $F \in \mathcal{F}_{2n+1,k}$, and a labeling, L , of the even numbered columns of T with elements from $\{x, 1, -1\}$ such that

1. all bricks have even size,
2. for all but the final brick, F has an Υ -match starting at each column that is not the final column of its brick,
3. for the final brick, the last column is empty and F has an Υ -match starting at each column that is not one of the final two columns of that brick,
4. the final column of each brick is labeled with 1, and
5. each even numbered column that is not a final even numbered column of a brick is labeled with x or -1 .

We then define the weight of C , $w(C)$, to be the product of all the x labels in L and the sign of C , $sgn(C)$, to be the product of all the -1 labels in L . For example, if $n = 12$, $k = 2$, $\Upsilon = St_{22}$, and $T = (4, 2, 4, 2)$, then Figure 12 pictures such a filled labeled brick tabloid $C \in \mathcal{K}_{14,2}^{(2),\Upsilon}$ with $w(C) = x^2$ and $sgn(C) = -1$.

Thus

$$(k(2n+1))!\Gamma^{(2)}(p_{2n+2,\nu}) = \sum_{C \in \mathcal{K}_{2n+2,k}^{(2),\Upsilon}} sgn(C)w(C). \quad (40)$$

1		-1				1		x				1		x				1	
5	21	12	16	17	26	11	15	20	23	8	24	25							
3	10	1	4	7	14	6	9	18	19	2	13	22							

Figure 12: A $C \in \mathcal{K}_{14,2}^{(2),St_{22}}$.

At this point, we can follow the same set of steps as in theorem Theorem 7 to prove

$$(k(2n+1))!\Gamma^{(2)}(p_{2n+2,\nu}) = \sum_{F \in \mathcal{F}_{2n+1,k}^{(2)}} x^{\Upsilon\text{-mch}^{(2)}(F)}. \quad (41)$$

That is, we can define the weight-preserving sign-reversing involution $I : \mathcal{K}_{2n+2,k}^{(2),\Upsilon} \rightarrow \mathcal{K}_{2n+2,k}^{(2),\Upsilon}$ in exactly the same way as in Theorem 7 and use it to prove (41) because the fact that the last cell is empty does not effect any of the arguments that we used in Theorem 7.

Applying $\Gamma^{(2)}$ to the identity (15) and using (35) and (41), we obtain

$$\begin{aligned}
\sum_{n \geq 1} \Gamma^{(2)}(p_{n,\nu}) t^n &= \sum_{n \geq 0} \frac{t^{2n+2}}{(k(2n+1))!} \sum_{F \in \mathcal{F}_{2n+1,k}^{(2)}} x^{\Upsilon\text{-mch}^{(2)}(F)} \\
&= \frac{\sum_{n \geq 1} (-1)^{2n-1} \nu(2n) \Gamma^{(2)}(e_{2n}) t^{2n}}{1 + \sum_{n \geq 0} (-1)^{2n} \Gamma^{(2)}(e_{2n}) t^{2n}} \\
&= \frac{\sum_{n \geq 1} (-1)^{2n-1} \frac{(-1)^{2n-1} (x-1)^{n-1}}{(k(2n-1))!} full_{2n-1}^{\Upsilon} t^{2n}}{1 - \sum_{n \geq 1} \frac{(x-1)^{n-1} t^{2n}}{(2kn)!} full_{2n}^{\Upsilon}} \\
&= \frac{\sum_{n \geq 1} \frac{(x-1)^{n-1}}{(k(2n-1))!} full_{2n-1}^{\Upsilon} t^{2n}}{1 - \sum_{n \geq 1} \frac{(x-1)^{n-1} t^{2n}}{(2kn)!} full_{2n}^{\Upsilon}}.
\end{aligned}$$

Dividing the above equation by t will then yield

$$\sum_{n \geq 0} \frac{t^{2n+1}}{(k(2n+1))!} \sum_{F \in \mathcal{F}_{2n+1,k}^{(2)}} x^{\Upsilon\text{-mch}^{(2)}(F)} = \frac{\sum_{n \geq 1} \frac{(x-1)^{n-1} t^{2n-1}}{(k(2n-1))!} full_{2n-1}^{\Upsilon}}{1 - \sum_{n \geq 1} \frac{(x-1)^{n-1} t^{2n}}{(2kn)!} full_{2n}^{\Upsilon}}$$

which proves (33).

Putting $x = 0$ in (33) gives us the generating function for the number of Υ -alternating elements of $\mathcal{F}_{2n+1,k}$. That is, if Υ be a subset of $\mathcal{F}_{2,k}$, then

$$\sum_{n \geq 0} \frac{Alt_{2n+1}^{\Upsilon} t^{2n+1}}{(k(2n+1))!} = \frac{\sum_{n \geq 1} \frac{(-1)^{n-1} t^{2n-1}}{(k(2n-1))!} full_{2n-1}^{\Upsilon}}{1 + \sum_{n \geq 1} \frac{(-1)^n t^{2n}}{(2kn)!} full_{2n}^{\Upsilon}}. \quad (42)$$

4 The proof of Theorem 3.

In this section we provide arguments similar to those in [10, Sect. 4] to prove Theorem 3. That is, suppose that $\Upsilon \subseteq \mathcal{F}_{j,k}$. Recall that

$$N_k^{\Upsilon}(t) = \sum_{n \geq 0} \frac{t^n}{(kn)!} \sum_{F \in \mathcal{F}_{n,k}} x^{\Upsilon\text{-nlap}(F)}, \quad (43)$$

and

$$A_k^{\Upsilon}(t) = \sum_{n \geq 0} \frac{t^n}{(kn)!} A_{n,k}^{\Upsilon} \quad (44)$$

where $A_{n,k}^{\Upsilon}$ is the number of $F \in \mathcal{F}_{n,k}$ with no Υ -matches. Our goal is to prove that

$$N^{\Upsilon}(x, t) = \frac{A^{\Upsilon}(t)}{1 - x(1 + (\frac{t}{k!} - 1)A^{\Upsilon}(t))}. \quad (45)$$

Let $\mathcal{E}_{n,k}^{\Upsilon}$ denote the set of all $F \in \mathcal{F}_{n,k}$ such that F has exactly one Υ -match and it occurs at the end of F , i.e. the unique Υ -match in F starts at position $n - j + 1$. We then let $E_{n,k}^{\Upsilon} = |\mathcal{E}_{n,k}^{\Upsilon}|$ and

$$B_k^{\Upsilon}(t) = \sum_{n \geq 1} \frac{E_{n,k}^{\Upsilon} t^n}{(kn)!}. \quad (46)$$

Lemma 11. Suppose that $\Upsilon \subseteq \mathcal{F}_{j,k}$. Then $B_k^\Upsilon(t) = 1 + \left(\frac{t}{k!} - 1\right) A_k^\Upsilon(t)$.

Proof. Suppose that $F \in \mathcal{F}_{n,k}$. For any set $S \subseteq \{1, \dots, k(n+1)\}$ of size k , let F^S be the $k \times (n+1)$ array so that the last column consists of the elements of S in increasing order, reading from bottom to top, and the first n columns is a $k \times n$ array, G , on the elements of $\{1, \dots, k(n+1)\} - S$ such that $\text{red}(G) = F$. For example, if $S = \{4, 6, 9\}$ and $F \in \mathcal{F}_{4,3}$ pictured on the left in Figure 13, then F^S is pictured on the right.

F	4	8	11	12
	3	6	7	10
	1	2	5	9

F^S	5	11	14	15	9
	3	8	10	13	6
	1	2	7	12	4

Figure 13: An example of F^S .

It is easy to see that for any set $S \subseteq \{1, \dots, k(n+1)\}$ of size k , if F has no Υ -matches then either F^S has no Υ -matches or $F^S \in \mathcal{E}_{n+1,k}^\Upsilon$. It follows that

$$\binom{k(n+1)}{k} A_{n,k}^\Upsilon = A_{n+1,k}^\Upsilon + E_{n+1,k}^\Upsilon \quad (47)$$

If we multiply both sides of (47) by $\frac{t^{n+1}}{(k(n+1))!}$ and sum for $n \geq 0$, we get that

$$\frac{t}{k!} A_k^\Upsilon(t) = A_k^\Upsilon(t) - 1 + B_k^\Upsilon(t)$$

or that

$$B_k^\Upsilon(t) = 1 + \left(\frac{t}{k!} - 1\right) A_k^\Upsilon(t).$$

□

Suppose that $F \in \mathcal{F}_{n,k}$ and $\Upsilon\text{-nlap}(F) = i \geq 0$. One can read any such F from left to right making a cut right after the first Υ -match. Say the first Υ -match ends in column i_1 . Then starting at column $i_1 + 1$ one can continue reading left to right and make another cut after the first Υ -match that one encounters. Continuing on in this way, one obtains i fillings F_1, \dots, F_i that have exactly one Υ -match and that Υ -match occurs at the end of the filling and is possibly followed by a filling, F_{i+1} , with no Υ -matches. In terms of generating functions, this says that

$$\begin{aligned} N_k^\Upsilon(x, t) &= A_k^\Upsilon(t) + x B_k^\Upsilon(t) A_k^\Upsilon(t) + (x B_k^\Upsilon(t))^2 A_k^\Upsilon(t) + \dots \\ &= \frac{A_k^\Upsilon(t)}{1 - x B_k^\Upsilon(t)}. \end{aligned} \quad (48)$$

Thus (45) immediately follows by substituting the expression for $B_k^\Upsilon(t)$ given in Lemma 11 into (48).

5 Computing $full_n^\Upsilon$

In this section, we show how to compute $full_n^\Upsilon$ for various $\Upsilon \subseteq \mathcal{F}_{2,k}$.

In the case $k = 2$, we can compute $full_n^\Upsilon$ for all Υ that are subsets of the set of standard tableaux of shape 2^2 . We clearly have only 3 choices for Υ in that case, namely, $\Upsilon_0 = \{P_1^{(2,2)}\}$ and $\Upsilon_1 = \{P_2^{(2,2)}\}$, and $\Upsilon_2 = \{P_1^{(2,2)}, P_2^{(2,2)}\}$, where $P_1^{(2,2)}$ and $P_2^{(2,2)}$ are pictured in Figure 4.

For Υ_0 , it is easy to see that $F \in Full_n^{\Upsilon_0} = Full_n^{P_1^{(2,2)}}$ if and only if $F(1, i) = 2i - 1$ and $F(2, i) = 2i$ for all i so that $full_n^{P_1^{(2,2)}} = 1$ for all $n \geq 2$. It is also easy to see that $F \in Full_n^{\Upsilon_2}$ if and only if F is a standard tableau of shape n^2 , so by the hook formula for the number of standard tableaux of shape 2^2 ,

$$full_n^{\Upsilon_2} = \frac{(2n)!}{n!(n+1)!} = \frac{1}{n+1} \binom{2n}{n} = C_n$$

where C_n is n -th Catalan number.

For Υ_1 , we have the following theorem.

Theorem 12. *For all $n \geq 2$, $full_n^{P_2^{(2,2)}} = C_{n-1}$, where $C_n = \frac{1}{n+1} \binom{2n}{n}$ is the n -th Catalan number.*

Proof. We shall give a bijective proof of this theorem by constructing a bijection from \mathcal{D}_{n-1} , which is the set of Dyck paths of length $2(n-1)$, onto $Full_n^{P_2^{(2,2)}}$. A Dyck path of length $2n$ is a path that starts at $(0,0)$ and ends at $(2n,0)$ and consists of either *up-steps* $(1,1)$ or *down-steps* $(1,-1)$ in such a way that the path never goes below the x -axis. It is well known that the number of Dyck paths of length $2n$ is equal to n -th Catalan number, $C_n = \frac{1}{n+1} \binom{2n}{n}$.

Now suppose that $F \in Full_n^{P_2^{(2,2)}}$. Because there is a $P_2^{(2,2)}$ -match starting in column i for $i = 1, \dots, n-1$, it follows that $F(1, i) < F(1, i+1) < F(2, i) < F(2, i+1)$ for $i = 1, \dots, n-1$. Thus $F(1, 1) < \dots < F(1, n)$ and $F(2, 1) < \dots < F(2, n)$. Now since $F(1, 1) < F(1, 2) < F(2, 1) < F(2, 2)$ and $F(n-1, 1) < F(n, 1) < F(2, n-1) < F(2, n)$, it follows that we must have that $F(1, 1) = 1$, $F(1, 2) = 2$, $F(2, n-1) = 2n-1$, and $F(2, n) = 2n$.

Let \mathcal{R}_n be the set of fillings, F , of the $2 \times n$ rectangle with the integers $1, \dots, 2n$ such that if the elements of F in the first row, reading from left to right, are a_1, \dots, a_n , and the elements of F in the second row, reading from left to right, are b_1, \dots, b_n , then (i) $a_1 < \dots < a_n$, (ii) $a_1 = 1$ and $a_2 = 2$, (iii) $b_1 < \dots < b_n$, and (iv) $b_{n-1} = 2n-1$ and $b_n = 2n$. Let \mathcal{E}_{n-1} be the set of paths of length $2(n-1)$ that consist of either *up-steps* $(1,1)$ or *down-steps* $(1,-1)$ such that the first step goes from $(0,0)$ to $(0,1)$ and the last step goes from $(2n-3, 1)$ to $(2n-2, 0)$. Here we do not require that the paths in \mathcal{E}_{n-1} stay above the x -axis.

There is a simple bijection, Θ , which maps \mathcal{E}_{n-1} onto \mathcal{R}_n . Given a path $P = (p_1, \dots, p_{2n-2}) \in \mathcal{E}_{n-1}$, we label the segments p_2, \dots, p_{2n-3} with the numbers $3, \dots, 2n-2$, as pictured in Figure 14. We then create a filling, $F = \Theta(P) \in \mathcal{R}_n$, by processing the elements $i \in \{3, \dots, 2n-3\}$ in order by putting i in the first available cell in the first row of F if p_{i-1} is an up-step and putting i in the first available cell in the second row of F if p_{i-1} is a down-step. See Figure 14 for an example.

It is easy to see that $\Theta(P)$ will be an element of \mathcal{R}_n since there are always $n-2$ up steps and $n-2$ down steps in p_2, \dots, p_{2n-3} . Clearly, the process is reversible, so Θ is a bijection.

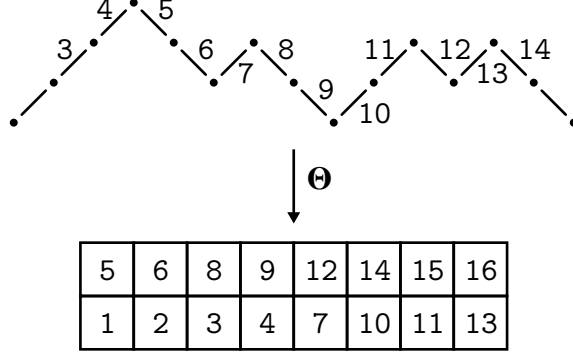


Figure 14: An example of the map Θ .

Thus all we have to prove is that $P \in \mathcal{D}_{n-1}$ if and only if $\Theta(P) \in \text{Full}_n^{P_2^{(2,2)}}$. Now suppose that $P = (p_1, \dots, p_{2n-2}) \in \mathcal{D}_{n-1}$ and $F = \Theta(P)$. We know that $F(1, 1) = 1$, $F(1, 2) = 2$ and $2 < F(2, 1) < F(2, 2)$. Thus there is a $P_2^{(2,2)}$ -match that starts in the first column of F . Now fix $2 \leq j \leq n-1$. We shall show that there is a $P_2^{(2,2)}$ -match starting in column j of $F = \Theta(P)$. Suppose that p_i is the j -th up-step of P , reading from left to right. Thus, $F(1, j+1) = i$ and the right-hand endpoint of p_i must have height at least 1, so there can be at most $j-1$ down-steps among p_1, \dots, p_i . Thus, no more than $j-1$ cells of the second row of F are filled with numbers less than i , while the first $j+1$ cells of the first row of F are filled with numbers that are less than or equal to i . This means that $F(2, j-1) < F(1, j+1) = i < F(2, j)$ which implies that $F(1, j) < F(1, j+1) < F(2, j) < F(2, j+1)$. Hence there is a $P_2^{(2,2)}$ -match starting in column j of F . Thus $\Theta(P) \in \text{Full}_n^{P_2^{(2,2)}}$.

Next suppose that $P = (p_1, \dots, p_{2n-2}) \in \mathcal{E}_{n-1} - \mathcal{D}_{n-1}$. Then P does not stay above the x -axis, so there is a smallest i such that the right-hand endpoint of p_i is at level -1 . It follows that for some $k \geq 2$, there must be k down-steps and $k-1$ up-steps in p_1, \dots, p_i , so $i = 2k-1$. This means $F(2, k) = i+1 = 2k$ and the elements $1, \dots, 2k$ occupy the first k columns of $F = \Theta(P)$. But this means that $F(1, k+1) > F(2, k)$, so there is not a $P_2^{(2,2)}$ -match starting in column k of F .

Thus the map Θ restricts to a bijection between \mathcal{D}_{n-1} and $\text{Full}_n^{P_2^{(2,2)}}$. Hence $\text{full}_n^{P_2^{(2,2)}} = C_{n-1}$ as claimed. \square

Our next goal is to classify those standard tableaux, $P \in St_{2k}$, such that $\text{full}_n^P = 1$. To this end, we say that a standard tableau, $P \in St_{2k}$, is *degenerate* if at least one of $P(i, 1) + 1 = P(i+1, 1)$ or $P(i, 2) + 1 = P(i+1, 2)$ holds for each $1 \leq i < k$. Then we have the following theorem.

Theorem 13. *Suppose $P \in St_{2k}$ is a standard tableau where $k \geq 2$. Then $\text{full}_n^P = 1$ for all $n \geq 1$ if and only if P is degenerate.*

Proof. Fix $P \in St_{2k}$. To help us visualize the order relationships within P , we form a directed graph G_P on the cells of the $k \times 2$ rectangle by drawing a directed edge from the position of the number i to the position of the number $i+1$ in P for $i = 1, \dots, 2n-1$. For example,

in Figure 15, the $P \in St_{25}$ pictured on the left, results in the directed graph G_P pictured immediately to its right. Then G_P determines the order relationships between all the cells in P since $P(r, s) < P(u, v)$ if there is a directed path from cell (r, s) to cell (u, v) in G_P . Now suppose that $F \in Full_n^P$ where $n \geq 3$. Because there is a P -match starting in column i , we can superimpose G_P on the cells in columns i and $i + 1$ to determine the order relations between the elements in the those two columns. If we do this for every pair of columns, i and $i + 1$ for $i = 1, \dots, n - 1$, we end up with a directed graph on the cells of the $k \times n$ rectangle which we will call $G_{P,n}$. For example, Figure 16, $G_{P,4}$ is pictured just to the right of G_P . It is then easy to see that if $F \in Full_n^P$ and there is a directed path from cell (r, s) to cell (u, v) in $G_{P,n}$, then it must be the case that $F(r, s) < F(u, v)$. Note that $G_{P,n}$ will always be a directed acyclic graph with no multiple edges.

In general, the problem of computing $full_n^P$ for a column strict tableau P of shape 2^k can be reduced to finding the number of linear extensions of a certain poset. That is, we claim the graph $G_{P,n}$ induces a poset $\mathcal{W}_{P,n} = (\{(i, j) : 1 \leq i \leq k \text{ \& } 1 \leq j \leq n\}, <_W)$ on the cells of the $k \times n$ rectangle by defining $(i, j) <_W (s, t)$ if and only if there is path from (i, j) to (s, t) in $G_{P,n}$. It is easy to see that $<_W$ is transitive. Since $G_{P,n}$ is acyclic, we can not have $(r, s) <_W (r, s)$. Thus $\mathcal{W}_P(n)$ is poset. It follows that there is a 1:1 correspondence between the elements of $Full_n^P$ and the linear extensions of $\mathcal{W}_{P,n}$. That is, if $F \in Full_n^P$, then it is easy to see that $(a_1, b_1), \dots, (a_{kn}, b_{kn})$ where $F(a_i, b_i) = i$ is a linear extension of $\mathcal{W}_{P,n}$. Vice versa, if $(a_1, b_1), \dots, (a_{kn}, b_{kn})$ is a linear extension of $\mathcal{W}_{P,n}$, then we can define F so that $F(a_i, b_i) = i$ and it will automatically be the case that $F \in Full_n^P$. Since every poset has at least one linear extension, it follows that $Full_n^P \neq \emptyset$ for all $n \geq 2$. Thus $full_n^P \geq 1$ for all $n \geq 1$.

Our next goal is to apply a simple lemma on directed acyclic geraphs with no multiple edges to replace $G_{P,n}$ by a simpler acyclic directed graph which contains the same information about the relative order of the elements in F . Given a directed acyclic graph $G = (V, E)$ with no multiple edges, let $Con(G)$ equal the set of all pairs $(i, j) \in V \times V$ such that there is a directed path in G from vertex i to vertex j . Then we have the following.

Lemma 14. *Let $G = (V, E)$ be a directed acyclic graph with no multiple edges. Let H be the subgraph of G that results by removing all edges $e = (i, j) \in E$ such that there is a directed path from i to j in G that does not involve e . Then $Con(G) = Con(H)$.*

Proof. The proof will be by induction on the number, k , of edges $e = (i, j) \in E$ such that there is a directed path from i to j in G that does not involve e . If $k = 0$, there is nothing to prove.

If $k > 0$, let $e = (i, j) \in E$ be an edge such that there is a directed path,

$$P = (i = v_1, v_2, \dots, v_s = j),$$

from i to j , in G , that does not involve e . Let H^* be the directed graph that results from G by removing edge e . Now suppose that $(x, y) \in Con(G)$. If there is a directed path from x to y in G that does not involve e , then clearly $(x, y) \in Con(H^*)$. Otherwise, suppose that $(x = w_1, \dots, w_t = y)$ is a directed path from x to y in G , for some $x, y \in V$, that uses the edge e . Thus for some $1 \leq r < t$, we have $w_r = i$ and $w_{r+1} = j$. But then $(w_1, \dots, w_r, v_2, \dots, v_{s-1}, w_{r+1}, \dots, w_t)$ is a path in H^* which connects x and y in H^* . Thus it follows that H^* is a connected acyclic graph with no multiple edges such that $Con(G) = Con(H)$. However, H^* has $k - 1$ edges, $f = (m, n) \in E - \{e\}$, such that there is a directed path from m to n in H^* that does not involve f . Thus by induction, we can remove all such edges from H^* to obtain an H with $Con(G) = Con(H^*) = Con(H)$. \square

Now suppose that we apply Lemma 14 to the graph $G_{P,4}$ in Figure 15 to produce $H_{P,4}$. We know that for any $((r,s),(u,v)) \in \text{Con}(G_{P,4})$, it must be the case that $F(r,s) < F(u,v)$ for all $F \in \text{Full}_4^P$. Thus since $\text{Con}(G_{P,4}) = \text{Con}(H_{P,4})$, it must be that case that whenever $((r,s),(u,v)) \in \text{Con}(H_{P,4})$, $F(r,s) < F(u,v)$ for all $F \in \text{Full}_4^P$. Now the directed edge $((1,2),(2,2))$ is not in $H_{P,4}$ since $G_{P,4}$ has the directed path $((1,2),(4,1),(5,1),(2,2))$. Also, directed edge $((3,2),(4,2))$ is not in $H_{P,4}$ because of the directed path $((3,2),(1,3),(4,2))$. Similarly, the corresponding edges in the third column of $G_{P,4}$ will be removed when creating $H_{P,4}$, so $H_{P,4}$ is the directed graph on the cells of the 5×4 rectangle pictured on the far right of Figure 15. Observe that in this case, $H_{P,4}$ has the property that outdegree of each vertex is 1 which means that $H_{P,4}$ determines a total order on the entries of any $F \in \text{Full}_4^P$. Thus, there is exactly one $F \in \text{Full}_4^P$, which means that $\text{full}_4^P = 1$ in this case.

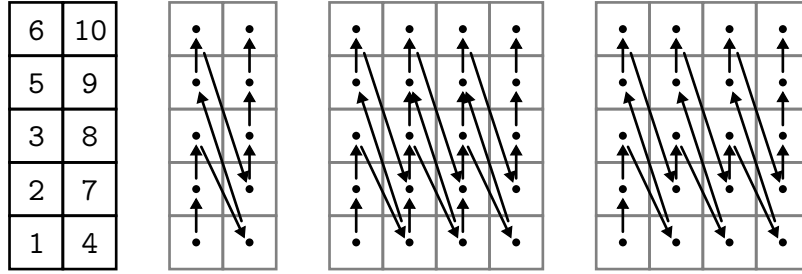


Figure 15: P , G_P , $G_{P,4}$, and $H_{P,4}$.

We should note, however, that it is not always be the case that applying Lemma 14 to remove edges from $G_{P,n}$ will produce a graph, $H_{P,n}$, where each vertex has outdegree 1. For example, consider the standard tableau Q pictured on the left in Figure 16. The graph G_Q is pictured immediately to the right of Q and the graph $G_{Q,4}$ is pictured immediately to the right of G_Q . The directed edge $((1,2),(2,2))$ is not in $H_{Q,4}$ due to the directed path of length 2, $((1,2),(4,1),(2,2))$, in $G_{Q,4}$. Also, the directed edge $((4,2),(5,2))$ is not in $H_{Q,4}$ since $G_{Q,4}$ has the directed path $((4,2),(2,3),(3,3),(5,2))$. Similarly, the corresponding edges in the third column of $G_{Q,4}$ are eliminated when creating $H_{Q,4}$ from $G_{Q,4}$. It is then easy to check that $H_{Q,4}$ is the directed graph on the cells of the 5×4 rectangle pictured on the far right of Figure 16. Note that in this case, cells $(3,3)$ and $(4,3)$ have outdegree 2. In fact, it is not difficult to see that there are exactly four $F \in \text{Full}_4^Q$ so that $\text{full}_4^Q = 4$ in this case.

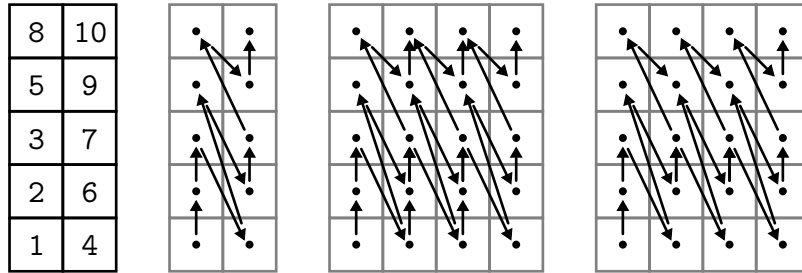


Figure 16: Q , G_Q , $G_{Q,4}$, and $H_{Q,4}$.

Let $P \in \mathcal{F}_{2,k}$ be a standard tableau. For all $n \geq 2$, let $H_{P,n}$ be the acyclic directed graph that arises from $G_{P,n}$ by removing all edges $e = ((r,s),(u,v))$ in $G_{P,n}$ such that there is a directed

path from (r, s) to (u, v) in $G_{P,n}$ that does not involve e . By Lemma 14, we know that $H_{P,n}$ imposes the same order relations on any $F \in \text{Full}_n^P$ as were imposed by $G_{P,n}$.

Suppose that P is a degenerate standard tableau in St_{2k} . We claim the outdegree of any vertex in $H_{P,n}$ is 1, which means that $H_{P,n}$ imposes a total order on the cells in the $k \times n$ rectangle. Thus, there is only one $F \in \text{Full}_n^P$ and, hence, $\text{full}_n^P = 1$ for all $n \geq 1$. Let $E = E(G_P)$ be the set of directed edges in the graph G_P and $E_n = E(G_{P,n})$ be the set of directed edges in $G_{P,n}$. First observe the out degree of any cell in column 1 or column n of $G_{P,n}$ is 1 since G_P is superimposed only once in column 1 and column n . However, for any column c with $2 \leq c \leq n-1$, G_P is superimposed on columns $c-1$ and c and on columns c and $c+1$ so the outdegree of each cell in column c could be 2. Next, observe that since P is a standard tableau, we always have that $P(i, 1) < P(i, 2)$ and the elements in the columns of P are strictly increasing from bottom to top. This means that there are three possibilities for an edge in E coming out of cell $(i, 1)$ for $i \leq n-1$, namely, it can go to cell $(i+1, 1)$, to cell $(i, 2)$, or to some cell $(g, 2)$, where $g < i$. Similarly there are only two possibilities for an edge in E coming out of cell $(i, 2)$ for $i \leq n-1$, namely, it can go to cell $(i+1, 2)$ or it can go to cell $(j, 1)$, where $j > i$. Since P is degenerate, we must have at least one of the edges $((i, 1), (i+1, 1))$ and $((i, 2), (i+1, 2))$ in G_P , so in fact, we only have 4 possible cases for pairs of edges that leave cells $(i, 1)$ and $(i, 2)$, as pictured in Figure 17.

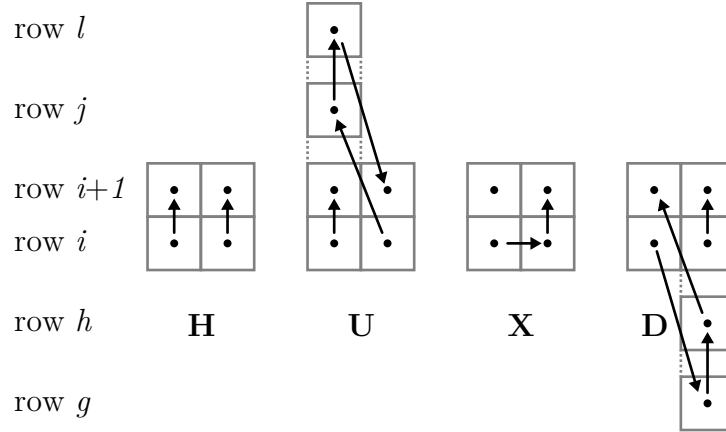


Figure 17: Possible edges in G_P for a degenerate standard tableau, P .

Case H. Edges $((i, 1), (i+1, 1))$ and $((i, 2), (i+1, 2))$ are in G_P .

In this case, the outdegree of cell (i, u) in column u of $H_{P,n}$ is 1 because $((i, u), (i+1, u))$ is the only edge out of each (i, u) in $G_{P,n}$.

Case U. Edges $((i, 1), (i+1, 1))$ and $((i, 2), (j, 1))$ are in G_P for some $j > i+1$.

In this case, since $P(i, 2) + 1 = P(j, 1)$ and $P(i+1, 2) > P(i, 2)$, G_P contains the path, $((i, 2), (j, 1), \dots, (l, 1), (i+1, 2))$, for some $l \geq j$, which passes through all the cells in the first column from $(j, 1)$ to $(l, 1)$. Now consider $G_{P,n}$, as shown in Figure 18. For each interior column, $2 \leq u \leq n-1$, $G_{P,n}$ contains the paths $((i, u), (i+1, u))$ and $((i, u), (j, u-1), \dots, (l, u-1), (i+1, u))$. Therefore, $H_{P,n}$ omits the edge $((i, u), (i+1, u))$ in all but its first column, leaving the single edge, $((i, u), (j, u-1))$, exiting each cell in row i .

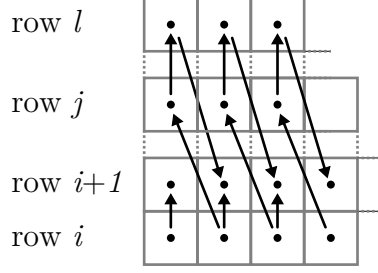


Figure 18: Edges in $G_{P,n}$ in Case U.

Case X. Edges $((i, 1), (i, 2))$ and $((i, 2), (i + 1, 2))$ are in G_P .

This case never occurs. Since P increases along rows and columns, this case would require $P(i, 1) < P(i + 1, 1) < P(i + 1, 2)$ while maintaining $P(i, 1) + 2 = P(i, 2) + 1 = P(i + 1, 2)$, which does not have an integer solution unless $P(i + 1, 1) = P(i, 2)$, which is impossible in P .

Case D. Edges $((i, 1), (g, 2))$ and $((i, 2), (i + 1, 2))$ are in G_P for some $g < i$.

In this case, since $P(i, 1) + 1 = P(g, 2)$ and $P(i + 1, 1) > P(i, 1)$, G_P contains the path, $((i, 1), (g, 2), \dots, (h, 2), (i + 1, 1))$, for some $g \leq h < i$, which passes through all the cells in the second column from $(g, 2)$ to $(h, 2)$. Now consider $G_{P,n}$, as shown in Figure 19. For each interior column, $2 \leq u \leq n - 1$, $G_{P,n}$ contains the paths $((i, u), (i + 1, u))$ and $((i, u), (g, u + 1), \dots, (h, u + 1), (i + 1, u))$. Therefore, $H_{P,n}$ omits the edge $((i, u), (i + 1, u))$ in all but its last column, leaving a single edge, $((i, u), (g, u + 1))$, exiting each cell in row i .

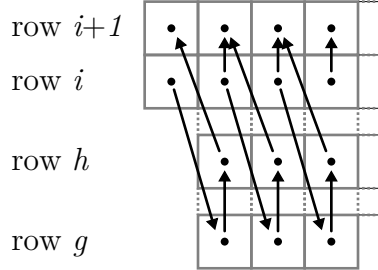


Figure 19: Edges in $G_{P,n}$ in Case D.

This shows that the outdegree of each vertex in $H_{P,n}$ is 1 as desired.

Now suppose that P is a standard tableau of shape 2^k that is not degenerate. We will show that $full_n^P \geq 2$. Let $i < k$ be the smallest i such that neither $P(i, 1) + 1 = P(i + 1, 1)$ nor $P(i, 2) + 1 = P(i + 1, 2)$. Thus, G_P has the edges $((i, 1), (g, 2))$ and $((i, 2), (j, 1))$, where $g \leq i < j$. Fix $n \geq 3$. We know $Full_n^P \neq \emptyset$ so that we fix some $F \in Full_n^P$.

Since F is column strict, all the elements in the second column below cell $(i, 2)$ are less than $y = F(i, 2)$ and all the elements in the second column above cell $(i + 1, 2)$ are greater than $z = F(i + 1, 2)$. That is, we have the situation pictured in Figure 20.

Let S be the elements in the first column of F which are in the interval (y, z) and let T be the elements of F in columns $3, \dots, n$ which are in the interval (y, z) . Since F has a P -match in the first two columns, we know that $F(j, 1)$ must be an immediate successor of $F(i, 2) = y$ relative to the elements of the first two columns of F so that $y < F(j, 2) < z$. Similarly, since

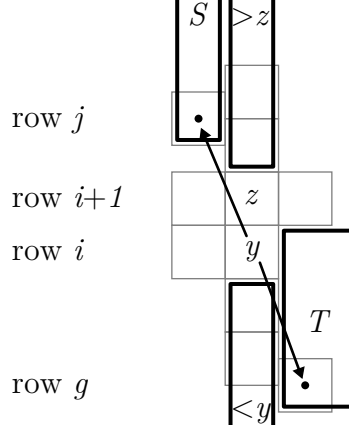


Figure 20: $G_{P,n}$ when P is not degenerate.

F has a P -match in columns 2 and 3, we know that $F(g, 3)$ must be an immediate successor of $F(i, 2) = y$ relative to the elements of in columns 2 and 3 of F so that $y < F(g, 3) < z$. Thus we know that S and T are non-empty. Let $s = |S|$ and $t = |T|$. It follows that the interval (y, z) consists of s elements in the first column and t elements in columns $3, \dots, n$ of F . Next suppose that we pick any set of S_1 of s elements of from $S \cup T$. We can then we modify the positions of the elements in the interval (y, z) in F by placing the elements of S_1 in column 1 in the positions that were occupied by S in column 1 of F in the same relative order as the elements of S in column 1 of F and placing the elements of $T_1 = (S \cup T) - S_1$ in the positions that were occupied by T in columns $3, \dots, n$ of F in the same relative order as the elements of T in columns $3, \dots, n$ of F . Call this modified filling F_{S_1} . Because we only changed the positions of elements in the interval (y, z) in F , it is easy to see that F_{S_1} will be an element of $Full_n^P$. Thus, $Full_n^P$ contains at least $\binom{s+t}{s}$ elements. Since $s, t \geq 1$, $\binom{s+t}{s} \geq 2$ and, hence, $full_n^P \geq 2$ for all $n \geq 3$. \square

Now that we have classified the standard tableaux, P , of shape 2^k such that $full_n^P = 1$ for all $n \geq 1$, we will determine the number of such column strict tableaux. Let $Deg_{2,k}$ be the set degenerate column strict tableaux of shape 2^k . A Motzkin path of length n is a lattice path on the integer lattice in the plane that runs from $(0, 0)$ to $(n-1, 0)$ consisting of up-steps $U = (1, 1)$, down-steps $D = (1, -1)$, and horizontal-steps $H = (1, 0)$ so that the path never passes below the x -axis. We let \mathcal{M}_n denote the set of Motzkin paths of length n . If we are allowed to color the horizontal-steps of the paths $M \in \mathcal{M}_n$ with one of k colors, then we obtain the set of k -colored Motzkin paths of length n . We let $\mathcal{M}_n^{(k)}$ denote the set of k -colored Motzkin paths. For $\mathcal{M}_n^{(2)}$, we shall denote the 2-colored horizontal-steps as H and \tilde{H} . It was proved in [6], [20] that $|\mathcal{M}_n^{(2)}| = C_{n+1}$ where C_n is the n -th Catalan number. By the hook formula, we know that $|St_{2^n}| = C_n$. Thus there exists a bijection from the set of standard tableaux of shape 2^n onto the set of 2-colored Motzkin paths of length $n-1$. We claim that we can define such a bijection $\Gamma : St_{2^n} \rightarrow \mathcal{M}_{n-1}^{(2)}$ for $n \geq 2$ as follows. Given a column strict tableau $P \in St_{2^n}$, define $\Gamma(P)$ to

be the path $M = (m_1, \dots, m_{n-1})$ such that m_i equals

$$\begin{cases} U & \text{if } G_P \text{ contains } (i, 1) \rightarrow (i+1, 1) \text{ and } (i, 2) \rightarrow (j, 1) \text{ for some } j > i+1, \\ D & \text{if } G_P \text{ contains } (i, 1) \rightarrow (g, 2) \text{ and } (i, 2) \rightarrow (i+1, 2) \text{ for some } g < i, \\ H & \text{if } G_P \text{ contains } (i, 1) \rightarrow (i+1, 1) \text{ and } (i, 2) \rightarrow (i+1, 2), \text{ and} \\ \tilde{H} & \text{if } G_P \text{ contains } (i, 1) \rightarrow (g, 2) \text{ and } (i, 2) \rightarrow (j, 1) \text{ for some } g \leq i \text{ and } j > i. \end{cases}$$

Note that we have not used the edge, e , out of cell $(n, 1)$ in this definition. We must first check that M is a 2-colored Motzkin path. This is equivalent to showing that (i) there are an equal number of U 's and D 's, i.e., the path ends at height 0, and (ii) we don't encounter more D 's than U 's in m_1, \dots, m_i for any i .

In order to verify condition (i), we notice that if $m_i = U$, both edges out of the cells in row i are directed toward the first column in G_P , whereas if $m_i = D$, both edges out of the cells in row i are directed toward the second column in G_P . If m_i equals H or \tilde{H} , then there is an edge directed toward each column out of the cells of row i in G_P . Thus if there are a U 's, b D 's, and c H 's and d \tilde{H} 's in M , then these account for $2a + c + d$ edges directed toward the first column in G_P and $2b + c + d$ edges directed toward the second column in G_P . The total in-degree of the first column is $n - 1$ since every entry except $(1, 1)$ has an edge of G_P coming into it. Similarly, the total in-degree of the second column is n . Since M accounts for every edge in G_P except e , which is directed towards the second column, it must be the case that $n - 1 = 2a + c + d = 2b + c + d$. Hence $a = b$ and there are an equal number of U 's and D 's in M . Thus M starts at $(0, 0)$ and ends at $(n - 1, 0)$.

If condition (ii) fails, then let i be the position of the first step to end beneath the x -axis, i.e. $m_i = D$ and there are an equal number of U 's and D 's among m_1, \dots, m_{i-1} . By our previous argument, there must be $i - 1$ edges directed into second column from the cells in rows $1, \dots, i - 1$. Since edges that go from the first column to the second column go weakly downward and edges from the second column to the second column can go up by at most one, it follows that the edges that hit the second column from cells in rows $1, \dots, i - 1$ must all end somewhere in the first i rows. Now suppose that $(g, 2)$, with $g \leq i$, is a cell that is not the endpoint of a directed edge from cells in rows $1, \dots, i - 1$. It cannot be that $g = i$ since this would force G_P to have edges $((i, 1), (i, 2))$ and $((i, 2), (i+1, 2))$. But this means that $P(i, 2) = 1 + P(i, 1)$ and $P(i+1, 2) = P(i, 1) + 2$ which is impossible since we must have $P(i, 1) < P(i+1, 1) < P(i+1, 2)$. Thus we must assume that $g < i$. Since m_i is a down-edge and the edge that goes into cell $(i, 2)$ can only have come from cell $(i-1, 2)$, it must be the case that G_P has edge $((i, 1), (g, 2))$. In this case, we must have the situation pictured in Figure 21. That is, because P is a column strict tableau, arrows that go from column 1 to column 2 cannot cross each other. Hence all the cells of the form $(h, 2)$ where $g < h \leq i$ must have been hit by vertical arrows. But this is impossible because then $\{P(i, 1), P(g, 2), P(g+1, 2), \dots, P(i, 2), P(i+1, 2)\}$ would be a set of consecutive elements so it would be impossible to have $P(i, 1) < P(i+1, 1) < P(i+1, 2)$, as is required for P to be a column strict tableau.

Since we know that $C_n = |St_{2^n}| = |\mathcal{M}_{n-1}^{(2)}|$, to prove that Γ is bijection, we need only show that Γ is 1:1. Let P_1 and P_2 be two distinct column strict tableaux of shape 2^n . Let s be the largest element such that $1, \dots, s$ are in the same cells in both P_1 and P_2 and suppose that s is in cell (i, b) . Then without loss of generality, we can assume that $s + 1$ is in column 1 in P_1 and in column 2 in P_2 . But this means that the arrow out of (i, b) goes to column 1 in G_{P_1} and to column 2 in G_{P_2} , which implies that the i -th step in $\Gamma(P_1)$ is not equal to the i -th step in $\Gamma(P_2)$, so $\Gamma(P_1) \neq \Gamma(P_2)$. Thus we have proved the following theorem.

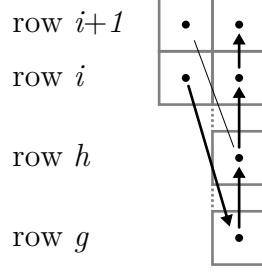


Figure 21: The case where $((i, 1), (g, 2)) \in E(G_P)$ and $g < i$.

Theorem 15. For all $n \geq 2$, $\Gamma : St_{2^n} \rightarrow \mathcal{M}_{n-1}^{(2)}$ is a bijection.

Note that $P \in St_{2^n}$ is degenerate if and only if there are no \tilde{H} 's in $\Gamma(P)$. Thus we have the following corollary.

Corollary 16. For all $n \geq 2$, $|Deg_n| = M_{n-1}$ where M_n is the number of Motzkin paths of length n .

We say that two column strict tableaux $P, Q \in \mathcal{F}_{2,k}$ are *consecutive-Wilf equivalent* (*c-Wilf equivalent*) if $A^P(t) = A^Q(t)$ and we say that P and Q are *strongly c-Wilf equivalent* if $D^P(t, x) = D^Q(t, x)$. Clearly, it follows from Theorem 2 that if P and Q are degenerate column strict tableaux of shape 2^k , then P and Q are strongly *c-Wilf equivalent*.

We end this section with a few remarks about how we have derived formulas for $full_n^P$ for certain column strict tableaux P . First there are some simple equivalences which help us reduce the number of $P \in St_{2^k}$ that we have to consider. Given any $F \in \mathcal{F}_{n,k}$, we define the generalized complement of F , F^{gc} as a composition of three operations on F , c for complement, f_y for flipping the diagram about the y -axis, and f_x for flipping the diagram about the x -axis. Here the operation of complement on F is the result of replacing each element i in F by $kn+1-i$. The operation f_y is the operation of taking a filling of the $k \times n$ rectangle with the numbers $1, \dots, kn$, and flipping the diagram about the y -axis and the operation f_x is the operation of taking a filling of the $k \times n$ rectangle with the numbers $1, \dots, kn$, and flipping the diagram about the x -axis. We then define $F^{gc} = f_x \circ f_y \circ c(F)$. This process is pictured in Figure 22.

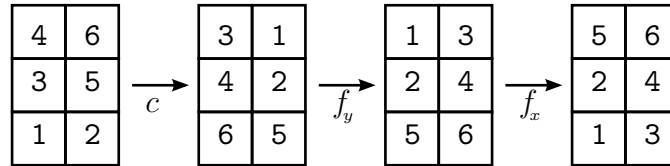


Figure 22: The generalized complement of a diagram.

It is easy to see that $F \in \mathcal{F}_{n,k}$ if and only if $F^{gc} \in \mathcal{F}_{n,k}$ and $P \in St_{2^k}$ if and only if $P^{gc} \in St_{2^k}$. Moreover for any column strict tableau P of shape 2^k , $F \in \mathcal{F}_{n,k}$ has a P -match in columns i and $i+1$ if and only if F^{gc} has a P^{gc} -match in columns $n-i$ and $n+1-i$. It follows that $F \in Full_n^P$ if and only if $F^{gc} \in Full_n^{P^{gc}}$ so that $full_n^P = full_n^{P^{gc}}$. Thus P and P^{gc} are *c-Wilf equivalent*. Now in $\mathcal{F}_{2,3}$, the only non-degenerate column strict tableaux that are not complements of each other are the tableaux P and Q pictured in Figure 23.

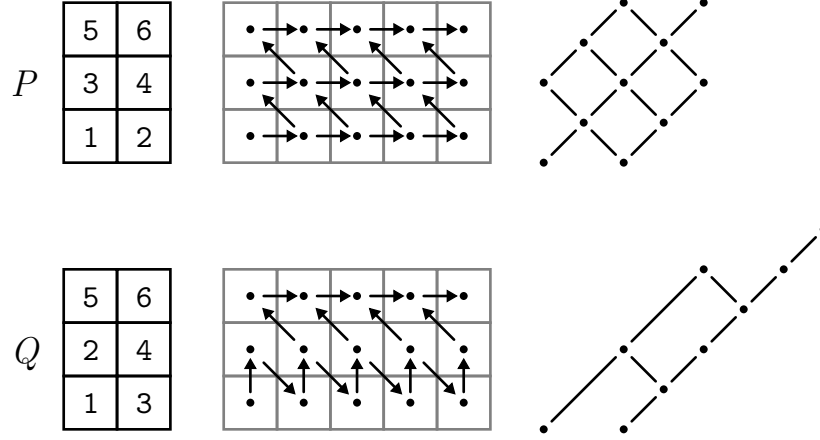


Figure 23: The tableaux P and Q and their corresponding Hasse diagrams.

Recall that in the proof of Theorem 13, we observed that for any standard tableaux $P \in St_{2k}$, the graph $G_{P,n}$ induces a poset $\mathcal{W}_{P,n} = (\{(i, j) : 1 \leq i \leq k \text{ \& } 1 \leq j \leq n\}, <_W)$ on the cells of the $k \times n$ rectangle by defining $(i, j) <_W (s, t)$ if and only if there is path from (i, j) to (s, t) in $G_{P,n}$ and that there is a one-to-one correspondence between the linear extensions of $\mathcal{W}_{P,n}$ and the elements of $Full_n^P$.

Now in the case of the standard tableaux P and Q in Figure 23, we have pictured $G_{P,5}$ just to the right of P and $G_{Q,5}$ just to the right of Q . In the case of P , if we eliminate the vertices corresponding to $(1,1)$ and $(1,2)$, which must always be the first two elements in any linear extension of $\mathcal{W}_P(5)$, and we eliminate the vertices in cells $(3,4)$ and $(3,5)$, which must always be the last two elements in any linear extension of $\mathcal{W}_{P,5}$, we are left with the Hasse diagram pictured to its right. In the case of Q , if we eliminate the vertices corresponding to cells $(1,1)$, $(2,1)$, $(1,2)$ and $(2,2)$, which must always be the first four elements in any linear extension of $\mathcal{W}_{Q,5}$, and we eliminate the vertices in cells $(3,4)$ and $(3,5)$, which must always be the last two elements in any linear extension of $\mathcal{W}_{Q,5}$, we are left with the Hasse diagram pictured to its right. We have developed general recursions of the number of linear extension of posets whose Hasse diagrams are similar to the one pictured for Q in Figure 23 which have allowed us to show that $full_n^Q = \frac{1}{2n+1} \binom{3n}{n}$. This work will appear in subsequent paper. However, we have not been able to find a formula for $full_n^P$.

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